Gaussian approach for phase ordering in nonconserved scalar systems with long-range interactions

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(Received 1 November 1993; revised manuscript received 5 July 1994)

We have applied the Gaussian auxiliary field method to a nonconserved scalar system with attractive long-range interactions, falling off with distance as $1/r^{d+\sigma}$, where d is the spatial dimension and $0 < \sigma < 2$. This study provides a test bed for the approach and shows some of the difficulties encountered in constructing a closed equation for the pair correlation function. For the relation $\phi = \phi(m)$ between the order parameter ϕ and the auxiliary field m, the usual choice of the equilibrium interfacial profile is made. The equation obtained for the equal-time two-point correlation function is studied in the limiting cases of small and large values of the scaling variable. A Porod regime at short distance and an asymptotic power-law decay at large distance are obtained. The theory is not, however, consistent with the expected growth law and attempts to retrieve the correct growth lead to inconsistencies. These results indicate a failure of the Gaussian assumption for this system, when used in the context of the bulk dynamics. This statement holds at least within the present form of the mapping $\phi = \phi(m)$, which appears to be the most natural choice, as well as the one consistent with the emergence of the Porod regime. By contrast, Ohta and Hayakawa have recently succeeded in implementing a Gaussian approach based on the interfacial dynamics of this system [Physica A **204**, 482 (1994)]. This clearly suggests that, beyond the simplicity of short-range "model A" dynamics, a Gaussian approach can only capture the essential physical features if the crucial role of wall motion in domain growth is explicitly considered.

PACS number(s): 64.60.Cn, 64.60.My

I. INTRODUCTION

The phase ordering dynamics of systems quenched from the disordered phase to the ordered phase has been extensively studied [1]. There is a general consensus that at the late stages of domain coarsening these systems enter a scaling regime [2], in which the equal-time, two-point correlation function has the scaling form

$$C(\mathbf{r},t) \equiv \langle \phi(\mathbf{x},t)\phi(\mathbf{x}+\mathbf{r},t)\rangle = f(r/L(t)) \; , \qquad (1)$$

where ϕ is the scalar order-parameter field, $L(t)$ is the characteristic length scale at time t after the the quench, f is a scaling function, and angular brackets indicate an average over initial conditions (and thermal noise, if present).

A first-principles calculation of the scaling function has proved to be a most difficult task. Even for the simplest model dynamics, that of a nonconserved order parameter (model A) [3] with purely short-ranged (SR) interactions, exact results are rare and available only for cases of limited physical interest [4].

In the past few years closed-approximation schemes for the two-point correlation function of the SR model A (SRMA) have been proposed by a number of authors [5–11], based on a mapping $\phi(\mathbf{r},t) = \phi(m(\mathbf{r},t))$ between the order parameter and an auxiliary field $m(\mathbf{r},t)$, where the zeros of $m(\mathbf{r},t)$ define the position of the domain walls. With this new variable the problem of describing the field at each instant of time is transformed into a problem of describing the evolution and statistics of the wall network. This approach enables the use of a physically plausible and mathematically convenient Gaussian distribution for m. Such a distribution is unacceptable for the order parameter field itself, since this is effectively discontinuous at the domain size scale.

The application of this sort of approach to both nonconserved and conserved (model B) dynamics, with purely SR interactions, has recently received a critical review by Yeung *et al.* [12]. Methods based on a description of the wall dynamics, first introduced by Ohta, Jasnow, and Kawasaki (OJK) [5], lead to an approximate *linear* equation for $m(\mathbf{r},t)$ or for its correlator $\langle m(\mathbf{x},t)m(\mathbf{x}+\mathbf{r},t)\rangle$. A different approach, due to Mazenko [7], aims at deriving a closed *nonlinear* equation for $C(\mathbf{r},t)$, built on the equation of motion for model A, using the single assumption that the field m is Gaussian distributed at all times. In this method the order parameter dynamics is taken into account everywhere and as the equation of motion is averaged over the initial conditions, the position of the walls is averaged over. For this reason, it is common to call it a "bulk approach." It has the following advantages: (i) the scaling functions have a nontrivial dependence on the spatial dimension d; (ii) the exponent λ, which characterizes the behavior of the different-time correlation functions, has a nontrivial value [10, 13], while it is simply $d/2$ within the usual interfacial methods; and (iii) it is also easily extensible to $O(n)$ component systems with topological defects, i.e., with $n \leq d$ [10].

The only uncontrolled feature of the approach is the Gaussian assumption. Recent simulation tests have shown, however, that this assumption is not entirely satisfactory: Blundell *et al.* [14] have made an absolute test (free of adjustable parameters) of the relation between two different scaling functions, revealing disappointing agreement with the theory. The discrepancy decreases, however, in higher dimensions, in agreement with an argument [11] that the Gaussian approximation becomes exact in the limit $d \to \infty$. Yeung *et al.* [12], using data of Shinozaki and Oono [15] for $d = 3$, have checked the single-point probability distribution for m, finding it to be flatter at the origin than a Gaussian. It is not difficult to derive an analytical expression for the *two-point* distribution $P(m(1), m(2))$, valid for $m(1)$, $m(2)$, and $|\mathbf{r}|$ small compared to $L(t)$ [16]. It differs from a Gaussian for fixed spatial dimension d, but is consistent with a Gaussian in the limit $d \to \infty$.

Despite these reservations, the Gaussian bulk approach has been shown to give good results for the SRMA, displaying most of the expected physical properties [7, 10]. In addition, there is very little difference between the predictions of OJK and Mazenko for the function $f(x)$, given by (1), at least for the SRMA with $d \geq 2$.

In this paper we test the validity of the Gaussian bulk approach, by extending it to model A dynamics with attractive long-range (LR) interactions. This application addresses a basic difficulty, not necessarily caused by the use of the Gaussian assumption: the attempts to extend the approach beyond the simplicity of the SRMA produce equations for $C(\mathbf{r}, t)$ which do not seem to respect the expected growth law for the typical domain size $L(t)$. A lack of a proper scaling of the terms in the equation for $C(\mathbf{r}, t)$, derived naively, is apparent for the case of a scalar order parameter, namely, for the LR model A (LRMA) and for the SR model B (SRMB), although not for a vector order parameter in which case a "naive" dimensional analysis of the equation agrees with the known growth law [17]. We shall see how this situation arises for the LRMA and present our understanding of it. In the case of the SRMB, a naive application of the method, however, omits the important bulk diffusion process which plays a vital role in the coarsening. Mazenko [8] has attempted, with limited success, to solve the problem by accounting explicitly for the bulk diffusion. The interesting feature of the LR systems, though, is that it is not immediately clear what, if any, physical process has been omitted or incorrectly treated in the present bulk approach. This observation becomes even more pertinent when one takes into account the success of a recent Gaussian interfacial approach, by Ohta and Hayakawa [18], where despite the LR interactions, sharp wall profiles are considered, an assumption which is also used in our work.

The scalar case, which is usually the more interesting one in the applications, is exceptional because an extra length, time independent at late stages, domain-wall thickness plays a role in the dynamics, and therefore power counting of lengths by dimensional analysis may not yield the right scaling (in terms of the characteristic length) for the different parts in the equation of motion. For the SRMB [19] and the LRMA [17, 20] dynamics the growth laws are $L(t) \sim t^{1/3}$ and $L(t) \sim t^{1/(1+\sigma)}$ (for $\sigma < 1$), respectively, for $n = 1$, and $L(t) \sim t^{1/4}$ and $L(t) \sim t^{1/\sigma}$, respectively, for $n > 2$ (with logarithmic corrections for $n = 2$ [17]), where $0 < \sigma < 2$ is the exponent describing the LR interactions, which decay as $1/r^{d+\sigma}$. For $n = 1$ and $1 < \sigma < 2$, the long-range interactions are irrelevant and the growth law is the same as that for the SR case [17, 20]. The SRMA, however, is exceptional since the predicted growth law $L(t) \sim t^{1/2}$ is the same for both the scalar and vector order parameters, accidently allowing for a naive dimensional analysis of the scalar equation of motion to agree with the growth law. In this case the role of the extra length in the scalar equation drops out as a result of two canceling errors [20]. Therefore we wonder if the success of the Mazenko method with this scalar model might be somewhat fortuitous. In other words, we raise the question of whether this approach (or any other closed bulk approach), naively applied, can succeed for those dynamic models where naive dimensional analysis gives the wrong growth law. In this respect it is interesting that a straightforward application of the method of Kawasaki, Yalabik, and Gunton (KYG) [6] to the LRMA [21] also gives the wrong growth law for $n = 1$, i.e., it gives the $t^{1/\sigma}$ growth suggested by naive dimensional analysis. This observation is consistent with our present line of thought, as the KYG method is not an interfacial approach.

In this paper we have developed an extension of Mazenko's approach for the LRMA. Just as in the application to systems with purely short-ranged interactions, the mapping function $\phi = \phi(m)$ has been taken to be the interface profile function. Besides being interesting in its own right, the study of this model provides a test bed for the approach and shows some of the difficulties that a bulk approximate theory must resolve.

II. THE MODEL WITH LONG-RANGE INTERACTIONS

We consider a system with long-ranged *attractive* interactions, falling off with distance as $r^{-(d+\sigma)}$. A suitable Hamiltonian functional of the scalar field is

$$H[\phi] = \int d^d r[(\nabla\phi)^2/2 + V(\phi)] + (J_{\rm LR}/2) \times \int d^d r \int d^d r' [\phi(\mathbf{r}) - \phi(\mathbf{r}')]^2 / |\mathbf{r} - \mathbf{r}'|^{d+\sigma} \ , \qquad (2)$$

where as usual we have taken the short-range part to have the Ginzburg-Landau form $J_{\rm LR} > 0$ and $V(\phi)$ has a local maximum at $\phi = 0$ and global minima at $\phi = \pm 1$. The model is well defined for $0 < \sigma < 2$. The equation of motion for a nonconserved field reads $\partial\phi/\partial t = -\delta H/\delta\phi$, i.e.,

$$\frac{\partial\phi(\mathbf{r}, t)}{\partial t} = \nabla^2\phi - V'(\phi) + V'_{\rm LR}(\phi) \ , \qquad (3)$$

where $V'(\phi) = dV/d\phi$ and the LR force is given, both in real and Fourier space, as

$$V'_{\rm LR}(\phi) = J_{\rm LR} \int d^d r' [\phi(\mathbf{r}') - \phi(\mathbf{r})]/|\mathbf{r} - \mathbf{r}'|^{d+\sigma} \qquad (4)$$

$$= J_{\rm LR}\, h(d, \sigma) \int \frac{d^d k}{(2\pi)^d}\, \phi(\mathbf{k})\ k^\sigma e^{i\mathbf{r}\cdot\mathbf{k}} \ , \qquad (5)$$

and

$$h(d,\sigma) = Q(d,\sigma)\ \frac{\sqrt{\pi}}{2^\sigma}\frac{\Gamma(-\frac{\sigma}{2})}{\Gamma(\frac{1+\sigma}{2})}\ , \tag{6}$$

$$Q(d,\sigma) = \pi^{\frac{d-1}{2}}\frac{\Gamma(\frac{1+\sigma}{2})}{\Gamma(\frac{d+\sigma}{2})}\ . \tag{7}$$

In (3) noise is absent since temperature is an irrelevant variable [22]. From an analysis of (3), assuming the validity of the scaling hypothesis (1), the following growth law has been predicted for a scalar order parameter [17, 20]:

$$\begin{aligned} L(t) &\sim t^{1/(1+\sigma)}\ , \quad 0<\sigma<1 \\ &\sim t^{1/2}\ , \quad 1<\sigma<2\ , \end{aligned} \tag{8}$$

in which the crossover $\sigma = 1$ separates the regime where domain growth is faster due to the LR correlations from the regime where these become irrelevant [17, 20].

III. THE SCALING EQUATION

To obtain an equation for the two-point correlation function (1) we multiply (3), evaluated at point $(1) \equiv (\mathbf{r}_1, t_1)$, by ϕ, evaluated at point $(2) \equiv (\mathbf{r}_2, t_2)$, and average over the ensemble of initial conditions yielding, at equal times,

$$\begin{aligned} \frac{1}{2}\frac{\partial C(1,2)}{\partial t} &= \nabla^2 C(1,2) - \langle \phi(2) V'(\phi(1))\rangle \\ &\quad + \langle \phi(2) V'_{\rm LR}(\phi(1))\rangle\ . \end{aligned} \tag{9}$$

We will call $\langle \phi(2) V'(\phi(1))\rangle$ and $\langle \phi(2) V'_{\rm LR}(\phi(1))\rangle$ the "nonlinear" (NL) and the "long-range" terms of the equation for $C(\mathbf{r},t)$, where $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$. In (9) the LR term reads, in both real and Fourier space,

$$\langle \phi(2) V'_{\rm LR}(\phi(1))\rangle$$

$$= J_{\rm LR} \int d^d r' \, [C(\mathbf{r}',t) - C(\mathbf{r},t)]/|\mathbf{r}-\mathbf{r}'|^{d+\sigma} \tag{10}$$

$$= J_{\rm LR}\, h(d,\sigma) \int \frac{d^d k}{(2\pi)^d}\, C(\mathbf{k},t)\ k^\sigma e^{i\mathbf{r}\cdot\mathbf{k}}\ . \tag{11}$$

Assuming the existence of a late-time single-scaling regime, we expect $C(\mathbf{r},t)$ to take the scaling form (1), in terms of which (9) reads

$$\begin{aligned} -\frac{1}{2}\frac{\dot{L}}{L}\ x f' &= \frac{1}{L(t)^2}\left(f'' + \frac{d-1}{x} f'\right) - \langle \phi(2) V'(\phi(1))\rangle \\ &\quad + \langle \phi(2) V'_{\rm LR}(\phi(1))\rangle\ , \end{aligned} \tag{12}$$

where $x = r/L(t)$ is the scaling variable and $f' = df/dx$, etc. In the equation above $\dot{L}/L \sim 1/t$, if $L(t)$ grows as a power law. The LR term now reads

$$\langle \phi(2) V'_{\rm LR}(\phi(1))\rangle$$

$$= \frac{J_{\rm LR}}{L(t)^\sigma} \int d^d x' \, [f(x') - f(x)]/|\mathbf{x}-\mathbf{x}'|^{d+\sigma} \tag{13}$$

$$= \frac{J_{\rm LR}\, h(d,\sigma)}{L(t)^\sigma} \int \frac{d^d y}{(2\pi)^d}\, g(y)\, y^\sigma e^{i\mathbf{x}\cdot\mathbf{y}}\ , \tag{14}$$

where $g(y)$ is the Fourier transform of $f(x)$, and $y = kL(t)$.

From an analysis of (3) for an isolated, stationary, planar wall, we find that to leading order the equilibrium *planar* wall profile saturates as

$$1 - \phi^2(r) \sim \frac{J_{\rm LR}}{V''_0\ r^\sigma} \quad (r \to \infty), \tag{15}$$

where $V''_0 = (d^2V/d\phi^2)_{\phi^2=1}$ and r is the distance from the wall. Hence we expect that throughout the bulk region $|\phi|$ will be below saturation by an amount $\sim 1/L(t)^\sigma$. Even with this power-law decay we still expect there to be well-defined walls, with a time-independent "thickness" w, defined, for example, from (15) via $w^\sigma = J_{\rm LR}/V''_0$. Therefore, domain walls may be regarded as "sharp" at late times, when $L(t) \gg w$. It follows that Porod's law [23], $g(y) \sim A(d,\sigma)/y^{d+1}$ for $y \gg 1$, holds within the regime $kw \ll 1 \ll kL(t) \equiv y$ [corresponding to $w \ll r \ll L(t)$ in real space], in which case Eq. (14) yields, for the leading scaling behavior of the LR term, as $x \to 0$,

$$\begin{aligned} \langle \phi(2) V'_{\rm LR}(\phi(1))\rangle &= \frac{J_{\rm LR}\, h(d,\sigma)}{L(t)^\sigma}\left(\int \frac{d^d y}{(2\pi)^d}\, g(y)\, y^\sigma + \frac{A(d,\sigma)\, h(d,1-\sigma)}{(2\pi)^d}\, x^{1-\sigma} + \cdots\right),\ 0<\sigma<1 \\ &= \frac{J_{\rm LR}\, h(d,\sigma)}{L(t)^\sigma}\left(\frac{A(d,\sigma)\, h(d,1-\sigma)}{(2\pi)^d}\, \frac{1}{x^{\sigma-1}} + O(1)\right),\ 1<\sigma<2\ . \end{aligned} \tag{16}$$

This result will be exploited below to determine the amplitude $A(d,\sigma)$ of the Porod tail, within the Gaussian approximation.

IV. THE GAUSSIAN APPROXIMATION

In order to transform (9) or (12) into a closed equation we need to express the NL term as some approximate nonlinear function of $C(\mathbf{r},t)$. A key idea, exploited by several authors [5–11] within SR model A dynamics, is to employ a nonlinear mapping between the order parameter $\phi(\mathbf{r},t)$, which at the scale of $L(t)$ is effectively discontinuous near walls and an auxiliary "smooth" field $m(\mathbf{r},t)$, whose zeros define the wall network. This introduces the wall structure into the problem and allows the approximation to be implemented through the new field.

From the equation of motion (3) we can see that, just as in the SRMA, if the initial field satisfies $|\phi| \leq 1$ everywhere, then this condition will hold at all times, ensuring that a one-to-one mapping can be defined. For this model we have in mind, following the analogous treatment [7] for SR interactions, to identify the field $m(\mathbf{r}, t)$ at points $\mathbf{r}$ near domain walls as *the (signed) distance to the nearest wall* (along its local normal), with the sign of m being that of ϕ. This determines m uniquely when $m \ll L(t)$. To specify m everywhere in space, we define the function $\phi(\mathbf{r}, t) = \phi(m(\mathbf{r}, t))$ by extending the suggestion [7] of using the equilibrium *planar* domain wall profile function for an isolated wall, with m the coordinate normal to the wall, i.e., the function $\phi(m)$ is specified by the equation

$$0 = \frac{d^2\phi(m)}{dm^2} - V'(\phi(m)) + J_{\rm LR} \int d^{d-1}y \int_{-\infty}^{+\infty} dm' \frac{[\phi(m') - \phi(m)]}{[(m'-m)^2 + \mathbf{y}^2]^{\frac{d+\sigma}{2}}} , \qquad (17)$$

with boundary conditions $\phi(0) = 0$ and $\phi(m) \to \mathrm{sgn}\,(m)$ for $|m| \to \infty$. Using (17) and taking m to be a Gaussian field, we can evaluate the nonlinear term $\langle \phi(2) V'(\phi(1)) \rangle$ in Eq. (12). The details are given in the Appendix. The LR part of the NL term, which follows from the last term in (17), is given by [see Eq. (A7) in the Appendix]

$$F_{\rm NL}(1,2) \equiv Q(d,\sigma) \int_{-\infty}^{+\infty} ds \frac{J_{\rm LR}}{|s|^{1+\sigma}} \times \langle \phi(m(2))[\phi(m(1)+s) - \phi(m(1))] \rangle \qquad (18)$$

$$= \frac{J_{\rm LR}\, a(d,\sigma)}{S_0(1)^{\sigma/2}} \int_0^{\frac{\pi}{2}C} d\theta \sec^\sigma(\theta) , \qquad (19)$$

where

$$a(d,\sigma) = h(d,\sigma) \frac{2^{1+\sigma/2}\Gamma(\frac{1+\sigma}{2})}{\pi^{3/2}} . \qquad (20)$$

According to our identification of $m(\mathbf{r}, t)$ as a distance from the interface, we expect $S_0 \equiv \langle m^2 \rangle$ to have the scaling form $S_0 = L(t)^2$, which can be used along with (19) and (A5) to rewrite Eq. (12) for the scaling function in the form

$$-\frac{1}{2}\frac{\dot L}{L}\, x f' = \frac{1}{L(t)^2}\left[f'' + \frac{d-1}{x} f' + \frac{2}{\pi}\tan\left(\frac{\pi}{2} f\right)\right] + \frac{J_{\rm LR}}{L(t)^\sigma}\left(\int d^dx' \frac{[f(x') - f(x)]}{|\mathbf{x}-\mathbf{x}'|^{d+\sigma}} - a(d,\sigma)\int_0^{\frac{\pi}{2}f} d\theta \sec^\sigma(\theta) \right) . \qquad (21)$$

For $\sigma < 2$ the SR part in (21), scaling as $1/L^2$, is negligible compared to the LR part, scaling as $1/L^\sigma$, and can be ignored (but see the discussion in Sec. VI). Demanding that the left-hand side of (21) balance the terms of order $1/L^\sigma$ on the right-hand side requires $\dot L/L \sim 1/L^\sigma$, i.e., that $L(t) \sim t^{1/\sigma}$. Note that this disagrees with the expected form (8). In Sec. V we will argue that a resolution of this discrepancy requires us to drop the left-hand side of (21) in leading order. For the moment, however, we pursue the original (and *a priori* natural) assumption that the left-hand side scales as $1/L^\sigma$ and write

$$L(t) = (J_{\rm LR}\, \mu\, t)^{1/\sigma} , \qquad (22)$$

where μ is to be determined. Dropping the SR terms from (21) gives the final equation for the scaling function $f(x)$:

$$0 = (\mu/2\sigma)\, x f' + \int d^dx' \frac{[f(x') - f(x)]}{|\mathbf{x}-\mathbf{x}'|^{d+\sigma}} - a(d,\sigma)\int_0^{\frac{\pi}{2}f} d\theta \sec^\sigma(\theta) . \qquad (23)$$

Equation (23) has to be solved numerically for general scaling variable x. However, it is straightforward to derive analytically the behavior for small and large x. Using the Porod's law form $f(x) = 1 - \alpha(d,\sigma)x + \cdots$ for small x (it is simple to show that this is the only consistent short-distance behavior), we find that the LR part of the NL term, given by (19), has a leading scaling behavior as $x \to 0$, which is similar to (16)

$$F_{\rm NL}(1,2) = \frac{J_{\rm LR}\, a(d,\sigma)}{L(t)^\sigma}\left(\frac{B(\frac{1-\sigma}{2}, \frac{1-\sigma}{2})}{2^{1+\sigma}} - \left(\frac{\pi\alpha}{2}\right)^{1-\sigma} \frac{x^{1-\sigma}}{1-\sigma} + \cdots \right) , \quad 0<\sigma<1$$
$$= \frac{J_{\rm LR}\, a(d,\sigma)}{L(t)^\sigma}\left[\left(\frac{\pi\alpha}{2}\right)^{1-\sigma} \frac{1}{x^{\sigma-1}(\sigma-1)} + O(1) \right] , \quad 1<\sigma<2 , \qquad (24)$$

where $B(x,y)$ is the beta function. Performing a small-x expansion of Eq. (23), we find that the dominant terms for $x \to 0$ are obtained from the terms multiplying $J_{\rm LR}$ in (21), whose small-x expansions are given by (16) and (24). Matching powers of x for general $0<\sigma<2$ and using $A(d,\sigma) = -\alpha(d,\sigma)(2\pi)^d/h(d,1)$ in (16) (which follows from Fourier transforming the Porod tail [24]), we find

$$\alpha(d,\sigma) = \frac{\sqrt{2}}{\pi}\left(\frac{\Gamma(\frac{d+1-\sigma}{2})}{\Gamma(\frac{d+1}{2})} \right)^{1/\sigma} , \quad 0<\sigma<2 \qquad (25)$$

for the coefficient of x in the small-x expansion of $f(x)$ [25]. For $\sigma = 2$, this reduces to the SR result $\alpha(d,2) = 2/(\pi\sqrt{d-1})$.

For $\sigma < 1$, (25) was obtained by matching the terms of $O(x^{1-\sigma})$ in (16) and (24). The leading (constant) terms yield an interesting sum rule to be satisfied by the structure factor scaling function $g(y)$ for this range of σ:

$$\int \frac{d^dy}{(2\pi)^d}\, g(y)\, y^\sigma = \frac{2^{\sigma/2}}{\Gamma(\frac{2-\sigma}{2})\, \sin(\frac{1+\sigma}{2}\pi)} , \quad 0<\sigma<1 . \qquad (26)$$

We now look at the large-x asymptotic form of Eq. (23) and discuss the large-x behavior of $f(x)$. In this limit, $f(x) \to 0$ and the final two terms in (23) become $g(0)/x^{d+\sigma}$ and $-a(d,\sigma)\,\pi f(x)/2$, respectively. In this regime, (23) can be integrated to give

$$f(x) \to \frac{2\sigma g(0)}{[(d+\sigma)\mu - \pi|a|\sigma]}\,\frac{1}{x^{d+\sigma}} + \frac{A}{x^{\pi|a|\sigma/\mu}}\ , \qquad (27)$$

where we note from (20) and (6) that a is negative. In general, both terms in (27) will be present in the large-x solution. On physical grounds, however, we do not expect $f(x)$ to fall off with distance more slowly (in a power-law sense) than the underlying interactions, which decay as $r^{-(d+\sigma)}$. (An exception is when sufficiently long-range power-law spatial correlations are present in the $t=0$ state. This power law can then persist for general times [26]. Here, however, we consider only short-range correlations in the initial state.) We infer that either $\mu < \pi|a|\sigma/(d+\sigma)$, so that the second term in (27) is subdominant for large x, or $A=0$. The first possibility, however, implies that the coefficient of the (dominant) first term in (27) is negative [since $g(0) > 0$ by definition], i.e., $f(x)$ approaches zero from below, which also seems unphysical (and disagrees with numerical simulations [27]). We conclude that the only physically sensible possibility is that A vanishes in (27). This can, presumably, only happen for a special choice of μ, so the condition $A=0$ determines μ. This mechanism is very similar to that which determines μ for short-range interactions [7, 10]. Note that, if $f(x)$ is to approach zero from above for $x \to \infty$, (27) gives the inequality

$$\mu > \pi|a|\sigma/(d+\sigma)\ . \qquad (28)$$

A sum rule for μ can be obtained by integrating (23) over space:

$$\mu = \frac{2\sigma|a|}{dg(0)} \int d^dx \int_0^{\pi f/2} d\theta\, \sec^\sigma\theta\ .$$

Finally, it should be noted that the above analysis implicitly assumes that $g(0)$ is finite, i.e., that $f(x)$ decays faster than x^{-d}. In fact, the mathematical structure allows for $f(x) \sim x^{-p}$ with $p < d$ [28], but we reject this possibility on the physical grounds that we appealed to before, namely that, at least for initial states with only short-ranged spatial correlations, the scaling function should not decay with a smaller power than the underlying interactions.

V. TWO-TIME CORRELATIONS

The Gaussian approach can also be used to evaluate the two-time correlation function $C(\mathbf{r},t_1,t_2) = \langle\phi(\mathbf{x},t_1)\phi(\mathbf{x}+\mathbf{r},t_2)\rangle$ and, in particular, the autocorrelation function $A(t_1,t_2) = C(0,t_1,t_2)$. The calculation is simplest in the limit $t_2 \gg t_1$, when $C \to 0$ and the full nonlinear equation can be linearized, Fourier transformed, and explicitly integrated. In this regime the analog of (9) for two-time correlations reads, in Fourier space (dropping the SR term on the right-hand side),

$$\frac{\partial C_{\mathbf{k}}}{\partial t_2} = -J_{\rm LR}|h|k^\sigma C_{\mathbf{k}} + \frac{\pi|a|}{2\mu t}C_{\mathbf{k}}\ , \qquad (29)$$

where (22) has been used for $L(t)$. We integrate (29) forward from time αt_1, where $\alpha \gg 1$ ensures that the condition $t_2 \gg t_1$, required for the validity of (29), holds at all times. This gives

$$C_{\mathbf{k}}(t_1,t_2) = C_{\mathbf{k}}(t_1,\alpha t_1)\left(\frac{t_2}{\alpha t_1}\right)^{\pi|a|/2\mu} \times \exp\{-J_{\rm LR}|h|k^\sigma(t_2-\alpha t_1)\}\ . \qquad (30)$$

Using the scaling form $C_{\mathbf{k}}(t_1,\alpha t_1) = L_1^d g_\alpha(kL_1)$, where $L_1 = L(t_1)$, and summing over $\mathbf{k}$ for $t_2 \gg \alpha t_1$ gives the autocorrelation function

$$A(t_1,t_2) = \text{const} \times (L_1/L_2)^{d-\pi|a|\sigma/2\mu}\ , \qquad (31)$$

where const is clearly independent of α. The physical requirement that A *decrease* with increasing t_2 gives the inequality $\mu > \pi|a|\sigma/2d$, which is guaranteed by (28) for $d > \sigma$. The connection between the parameter μ and the exponent describing the decay of the autocorrelation function is similar to that obtained for purely short-ranged interactions [10, 13].

VI. DISCUSSION AND SUMMARY

We have extended the original Mazenko Gaussian approach [7] to the LR model and evaluated the late-time leading contribution to the NL term of Eq. (21), yielding a dominant LR part given by (19), which is of order $1/L^\sigma$. An infinitely sharp wall profile has been used, which amounts to neglecting a quantity of relative order $1/L^\sigma$. The LR term in Eqs. (11)–(14) is of the same order as the nonlinear term and has an amplitude which is a function of x, d and σ, but its nonlocal nature [i.e., its dependence on the values of $f(x)$ everywhere] makes the problem particularly hard to handle.

Despite the profile power-law decay (15) induced by the LR interactions, the scaling function exhibits Porod's law, i.e., a linear short-distance behavior in real space with the coefficient given by (25). This is consistent with the assumption that at late times there are well-defined walls with a constant "width" independent of $L(t)$. This is an important point of principle, on which the identification of the field $m(\mathbf{r},t)$ and the mapping (17) rely, and also a key ingredient in the first-principles derivation [17, 20] of the growth law (8). A proper decay of the autocorrelation function (31) is also found.

The central question we want to address in this paper is whether the Gaussian theory based on the bulk dynamics is able to yield the correct growth law for this model. We have seen that the naive application of the Gaussian approach presented in Sec. IV ostensibly gives the wrong growth law: (22) instead of (8). A related problem is the SRMB to which Mazenko has attempted to apply the Gaussian approach [8], yet the correct growth law does not come out of the theory as cleanly as in the SRMA. In this system local conservation imposes a bulk diffusion process which controls the interface motion and delays domain coarsening relative to the purely relaxational dynamics of model A. There are some common features between the dynamics of a conserved and a LR interacting field, namely, the existence of a bulk profile which relaxes rapidly to a nonsaturating value as the walls move. One key difference, though, is that the true growth law for the SRMB ($t^{1/3}$) is *faster* than that obtained by a naive application of the Gaussian approach (without allowing for bulk diffusion), which gives $t^{1/4}$.

Before implementing any approximation we focus the analysis on the exact equation (12) for the scaling function $f(x)$. If the growth law (8) holds, the time-derivative term must be negligible compared to the LR term (14), which scales as $1/L^\sigma$, and therefore the NL term must have a leading contribution of order $1/L^\sigma$, which exactly cancels the LR term in the scaling limit. In fact, this condition determines the late-time leading contribution to the scaling function. Within the Gaussian approximation, it amounts to neglecting the first term in (23) [which came from the left-hand side of (21)] to give

$$0 = \int d^dx' \frac{[f(x') - f(x)]}{|\mathbf{x} - \mathbf{x}'|^{d+\sigma}} - a(d,\sigma) \int_0^{\frac{\pi}{2} f} d\theta \sec^\sigma(\theta) \ . \tag{32}$$

Solving this equation gives the scaling function $f(x)$, within the Gaussian approach, provided the growth law is *slower* than $t^{1/\sigma}$. However, there seems to be no way to *determine* the growth law within this scheme. Moreover, (32) has a serious shortcoming. If we integrate the equation over all space, the first term drops out, giving the sum rule

$$\int_0^\infty dx\, x^{d-1} \int_0^{\pi f(x)/2} d\theta \, \sec^\sigma(\theta) = 0 \ . \tag{33}$$

Since the integrand is positive definite, the only way this sum rule can be satisfied is for $f(x)$ to be negative for some range (or ranges) of x, with sufficient negative weight to satisfy (33). This seems *a priori* improbable for a nonconserved order parameter and indeed numerical simulations [27] show no hint of it.

We emphasize, however, that since our fundamental equation (12) is *exact*, the analog of (33) obtained *without making the Gaussian approximation* must be exactly true. Because the true growth is *slower* than $t^{1/\sigma}$, the left-hand side of (12) is negligible in the scaling limit. Taking the Fourier transform of the equation and setting $k = 0$, the LR term vanishes. This leaves (to leading order) the identity

$$\int d^dx \, \langle \phi(2) V'(\phi(1)) \rangle = 0 \ , \tag{34}$$

of which (33) is the special case obtained within the Gaussian approximation.

A feature central of the present analysis is that in order to capture the essential role of wall dynamics in phase ordering process, an extra length scale, the typical wall width w must emerge naturally within the approach adopted. This argument follows from a scaling analysis of the equation of motion for the order parameter ϕ, which ultimately leads to a w dependence in the growth law $L(t)$ [17,20] (except the case when $L \sim t^{1/2}$, where w accidentally cancels out). Following our argument above, the exact NL term $\langle \phi(2) V'(\phi(1)) \rangle$ must provide a subleading contribution of order $w/L^{1+\sigma}$ for $0 < \sigma < 1$ and $w^{2-\sigma}/L^2$ for $1 < \sigma < 2$, which matches the time derivative in (12). This contribution is absent in (19) and the length scale w does not emerge (for instance, as a short-distance cutoff) at any stage of our calculation.

Our results seem to indicate that the Gaussian approach, applied to the bulk equation of motion, is unable to account for the qualitative features of coarsening in systems with long-range interactions. This statement holds, however, within our particular choice of the Gaussian field m, which is related to ϕ by the equilibrium interfacial profile, i.e., the solution of (17). Since we cannot exhaust all the possibilities, there is always a chance that there may exist a mapping definition which is physically more appropriate and works better than (17). For example, one can in principle use the same mapping $\phi''(m) = V'(\phi(m))$ as in the case of purely short range interactions. However, this leads to an inconsistent scaling analysis of the equation for $C(\mathbf{r}, t)$: at short distances there is no LR part in the NL term to match the LR term (16) and the Porod regime is lost; besides, $m(\mathbf{r}, t)$ can no longer be regarded as a distance from a wall. By contrast, the mapping employed here seems far more natural and physically suitable for a system with LR interactions. We summarize by saying that, beyond the simple nonconserved systems with SR interactions, one cannot apply the Gaussian bulk approach in a straightforward and naive manner to construct a closed equation for the scaling function $f(x)$, as it fails to yield a growth law different from that obtained from dimensional analysis of the linear terms in the equation of motion. The reason for this failure appears to be related to the inability of the bulk approach to capture the essential role of the wall dynamics in the domain coarsening. Just as for the conserved scalar system, a deeper understanding of the underlying physics is required in order to implement a more controlled approximate scheme. This is clearly suggested by the absence of the length scale w in our results and by the success of an interfacial approach for the same system, recently proposed by Ohta and Hayakawa [18].

Finally, we note that the present methods can also be used for a vector order parameter with long-range interactions. In that case the $t^{1/\sigma}$ growth obtained within the Gaussian approach is correct (apart from logarithmic corrections for $n = 2$ [17]). The purpose of the present paper, however, is to test the method on those systems which provide the greatest challenge, i.e., scalar systems, in the hope that the difficulties identified here may stimulate the development of more robust approximation schemes.

ACKNOWLEDGMENTS

A.B. thanks T. Ohta for a stimulating discussion. J.F. thanks JNICT (Portugal) for financial support.

APPENDIX

In this appendix we show how the nonlinear term in the scaled equation of motion [i.e., the last term in Eq. (21)] was obtained. Using (17), we can write

$$\langle\phi(2)V'(\phi(1))\rangle=\left\langle\phi(2)\frac{d^2\phi(m(1))}{dm(1)^2}\right\rangle+Q(d,\sigma)\int_{-\infty}^{+\infty}ds\frac{J_{\rm LR}}{|s|^{1+\sigma}}\langle\phi(m(2))[\phi(m(1)+s)-\phi(m(1))]\rangle \tag{A1}$$

$$=\int\frac{dk}{2\pi}\int\frac{dk'}{2\pi}\tilde\phi(k)\tilde\phi(k')\left[J_{\rm LR}\,h(d,\sigma)k^\sigma-k^2\right]\left\langle e^{im(1)k+im(2)k'}\right\rangle, \tag{A2}$$

where $h(d,\sigma)$ and $Q(d,\sigma)$ are given by (6) and (7) and $\tilde\phi(k)$ is the Fourier transform of $\phi(m)$.

Following Mazenko [7], we now make the key assumption that $m(\mathbf{r},t)$ is a Gaussian field (with zero mean) at all times, with a pair distribution function

$$P(m(1),m(2))=N\;\exp\Big[-\frac{1}{2(1-\gamma^2)}\Big(\frac{m(1)^2}{S_0(1)}+\frac{m(2)^2}{S_0(2)}-\frac{2\gamma\,m(1)m(2)}{\sqrt{S_0(1)S_0(2)}}\Big)\Big], \tag{A3}$$

$$S_0(1)=\langle m(1)^2\rangle,\quad \gamma(1,2)=\frac{\langle m(1)m(2)\rangle}{\sqrt{S_0(1)S_0(2)}},$$
$$N=\frac{1}{2\pi\sqrt{(1-\gamma^2)S_0(1)S_0(2)}}. \tag{A4}$$

We also note that, as the walls become effectively sharp in the late-time regime, we can use the profile $\phi(m)=\mathrm{sgn}\,(m)$ to evaluate the leading contribution to the scaling functions. From (15) we expect that the effect of ignoring the power tail in the profile is to neglect a quantity of relative order $\sim 1/L(t)^\sigma$ in the LR part of (A2). The purely SR part of the NL term is then simply given, as a nonlinear function of $C(\mathbf{r},t)$, by the result [7] for the SRMA

$$\left\langle\phi(2)\frac{d^2\phi(m(1))}{dm(1)^2}\right\rangle=-\frac{2}{\pi\,S_0(1)}\tan\left(\frac{\pi}{2}C\right). \tag{A5}$$

Deriving a similar result for the LR part of the NL term is more tricky. There are three different ways to perform the calculation: we will outline the basic steps of each one. Representing $\phi(m)$ in Fourier space and Taylor expanding $\phi(m{+}s)$ in powers of s, using the Gaussian property and returning to real space gives the formal expansion for the second term in (A1) [29] [which we denote by $F_{\rm NL}(1,2)$, according to definition (18)]

$$F_{\rm NL}(1,2)=Q(d,\sigma)\int_{-\infty}^{+\infty}ds\frac{J_{\rm LR}}{|s|^{1+\sigma}}\times\sum_{n=1}^{\infty}\frac{s^{2n}}{(2n)!}2^n\frac{\partial^nC(1,2)}{\partial S_0(1)^n}. \tag{A6}$$

Using $C(1,2)=\langle\mathrm{sgn}\,(m(1))\mathrm{sgn}\,(m(2))\rangle$, the integral representation $\mathrm{sgn}\,(m)=1/(i\pi)\int_{-\infty}^{+\infty}dz\exp[izm]/z$, and the Gaussian property, the series can be summed. Finally, differentiating with respect to $C_0(1,2)=\gamma\sqrt{S_0(1)S_0(2)}$, performing the z and s integrals, and integrating back yields the nonlinear function

$$F_{\rm NL}(1,2)=\frac{J_{\rm LR}\,a(d,\sigma)}{S_0(1)^{\sigma/2}}\int_0^{\frac{\pi}{2}C}d\theta\sec^\sigma(\theta), \tag{A7}$$

which is the result (19), with $a(d,\sigma)$ given by (20).

Alternatively, we can take $\phi(m)=\mathrm{sgn}\,(m)$ from the start and do the s integral, giving

$$F_{\rm NL}(1,2)=-\frac{2J_{\rm LR}Q(d,\sigma)}{\sigma}\left\langle\frac{\mathrm{sgn}\,(m(1))\,\mathrm{sgn}\,(m(2))}{|m(1)|^\sigma}\right\rangle, \tag{A8}$$

use integral representations for $\mathrm{sgn}\,m$ and $1/|m|^\sigma$, do the Gaussian integral, differentiate with respect to $C_0(1,2)$, perform the remaining integrals, and finally integrate back over $C_0(12)$, yielding the same result. Taking into account (15) and using a $\mathrm{sgn}\,m$ profile, (A8) can be recognized as the leading order result for a ϕ^4-potential NL term, i.e., $\langle\phi(2)\phi(1)[1-\phi^2(1)]\rangle$ [30]. Finally, the simplest derivation is to take the integral representation of $\mathrm{sgn}\,m$ in (A8), use the Gaussian property, differentiate with respect to $C_0(1,2)$, and do the Gaussian integral, leading to the same point as the first calculation before its final integrations. This derivation, however, does not provide the appealing intermediate expressions (A7) and (A8).

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